U. Yagit on Model categories I

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Def Given a diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$
a *lifting* is a map $h: B \to X$ ~~such that~~ $hi = f$ and $ph = g$

If $\exists h$ for any f and g, i has *left lifting property* w.r.t. p and p has *RLP* with respect to i.

Def A model category $\mathcal{C}$ is a cat with 3 classes of maps: weak equivs, cofibrations + fibrations.

A *trivial* (or acyclic) *fibration / cofib* if it is both a fib / cofib and a weak equiv

M satisfies 5 axioms

MC1 $\mathcal{C}$ has all small limits + colimits

MC2 If f and g are maps with fg defined, if two of f, g and fg are weak equivs, so is the third.

MC3 A retract of a weak equiv / fib / cofib is a weak equiv / fib / cofib.

MC4 Given a diagram
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$
, $\exists$ lifting h if i is cofib and p is triv fib or i is triv cofib and

i is fibration

MC5 Any map $X \xrightarrow{f} Y$ can be functorially factored in two ways

i) $f = pi$ with i is cofib, p is triv fib

ii) $f = pi$ with i a triv cofib, p is fib.

Can show each class of morphism is closed under composition and include all identity maps.

$\mathcal{C}$ has initial object ϕ and terminal object $*$. by MC1

Def X is cofibrant if $\phi \to X$ is cofibration
X is fibrant if $X \to *$ is fibration.

Example The category Top of topological spaces can be given an MC structure by defining $f: X \to Y$ where

i) weak equivs are weak htpy equivs

ii) fibrations are Serre fibrations
(Def. $p: X \to Y$ is a Serre fib if p has RLP w.r. to all map $A \to A \times I$ for CW cxs A)

iii) cofibrations are maps with LLP w.r. to all trivial fibrations.

Can do the same for pointed spaces].

A MC is determined by any two of its 3 classes of morphisms.

<u>Prop</u> Let $\mathcal{C}$ be a model category

1) A map $i : A \to B$ is a cofib iff it has LLP with respect any trivial fibration
2) It is trivial cofib it has LLP w.r.to all fibrations
3) dual to 1)
4) dual to 2)

Proof of 1) $\Rightarrow$ is MC4. For $\Leftarrow$ use MC5

$i = pj$ where p is acyclic fib and j is cofib

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "g"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A \arrow[r] \arrow[d, "j"] & A \arrow[d, "i"] \\ B \arrow[r, "g"] & Z \arrow[r, "p"] & B \end{tikzcd}$$

i is retract of the cofib j and is $\therefore$ a cofib.

QED

<u>Prop</u> Any 2 of the 3 classes of maps determine the third.

<u>Proof</u>: for fib + cofibs defined. We can define trivial fibs + cofibs by lifting property

A weak equiv is any composition of triv cofib followed by triv fibration QED

Can also show that classes of cofibrations + triv cofibrations determine the rest.

Def An object A in an ~~topological~~ cat $\mathcal{C}$ is **compact** if for any diagram

$$\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots \qquad n \in \mathbb{N}$$

$$\operatorname{colim}_n \mathcal{C}(A, X_n) \cong \mathcal{C}(A, \operatorname{colim} X_n)$$

Def A class of maps J **permits the small object argument** if $\mathcal{C}$ is complete and the domains of J are all compact.

Def A cofibrantly generated MC is a MC such that

(1) There is a set I of maps called generating cofibrations that permits the SOA and such that a map p is a trivial fib iff it has RLP w.r.

(2) There is a set J of maps called generating trivial cofibrations .... (as above)

Example $\mathcal{C} = \mathcal{T}op$

$I = \{ i_n : n \geq 0 \}$ where $i_n : S^{n-1} \hookrightarrow D^n$

$J = \{ j_n : n \geq 0 \}$ $\qquad j_n : I^n \hookrightarrow I^n \times I$

$LLP(RLP(I))$ is set of trivial cofibs

etc.

$\mathcal{C} = \mathcal{T}^G$ = pointed G-spaces + equiv maps

<u>Def</u> An equiv map $f : X \to Y$ is a <u>naive Serre fib</u> if it is a Serre fib in $\mathcal{T}$. It is a <u>genuine Serre fib</u> if $f^H : X^H \to Y^H$ is Serre fib for all $H \subseteq G$.

<u>Thm</u> In the naive case the set of gen cofibs is

$$I'_G = \{ i_{n+} \wedge G_+ : n \geq 0 \}$$
$$J'_G = \{ j_{n+} \wedge G_+ : n \geq 0 \}$$

In the genuine case

$$I_G = \{ i_{n+} \wedge G/H_+ : n \geq 0,\ H \subseteq G \}$$
$$J_G = \{ j_{n+} \wedge G/H_+ : \quad " \quad \}$$

These define two CGMC structures on $\mathcal{T}^G$.

A cofibration is a retract of a relative CW complex.